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Quantum Dynamics in the Fermi–Pasta–Ulam Problem

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We study a quantum chain of oscillators with nonlinear quartic interactions, under the "narrow packet" approximation. We analyse the dynamics of quantum corrections and the conditions at which the quantum solution for average complex amplitude converges to the corresponding classical unstable solution which describes the four-wave decay processes of phonons. We develop an asymptotic theory by using a small quasiclassical parameter, and determine the characteristic time scale for which the evolution of decay processes is essentially specified by quantum effects.

KEY WORDS: quantam chain of oscillators; wave decay processes; narrow packet approximation; Cauchy problem; Fuchs-type equations; asymptotics.

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1. INTRODUCTION

In a nonlinear environment with dispersion waves may be unstable under decay processes, see, for instance, Zakharov (1996). The instability is observed by effective interaction of waves with vectors $\vec{k}_j$ and frequencies $\omega(\vec{k}_j)$ in a neighbourhood of resonances

$$\begin{aligned} \sum_j n_j\, \omega(\vec{k}_j) &= 0, \\ \sum_j n_j\, \vec{k}_j &= 0, \end{aligned} \tag{1.1}$$

whence the amplitudes of the initially small waves change exponentially fast in time at the initial stage, and nonlinear effects turn out to be essential for describing

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their dynamics. In the Fermi–Pasta–Ulam (FPU) problem these decay processes represent the first step in developing a stochastic behavior of the system (Berman and Izrailev, 2005; Berman and Kolovskii, 1984).

The wave decay processes are of considerable interest in problems of Bose–Einstein condensates (BEC), chemistry, hydrodynamics of liquids and gases, plasma physics, nonlinear optics, solid physics, etc.

Usually the dynamics of wave decay processes is described in the framework of classical approach. Such approach seems to be justified so far the energy of interacting waves is sufficiently large and effects related to quantities of order $\hbar$ do not become transparent. However, it is not always possible to neglect the influence of quantum mechanical corrections on system dynamics even in the quasiclassical setting, if particularly the classical approximation is unstable. The study of dynamical stochasticity in classical and quantum mechanics shows that if the classical system is strongly unstable then its quantum dynamics may essentially differ from the classical one, see Berman and Zaslavskiis (1982), Berman *et al.* (2002), Zaslavskii (1984), Berman *et al.* (2004), Berman and Vishik (2003), etc.

The present paper is devoted to quantum mechanical analysis of the dynamics of decay processes of type (1.1) which occur in a one-dimensional nonlinear chain of connected oscillators, see Fermi *et al.* (1955). The Hamiltonian of the system has the form

$$H = \sum_{n=1}^{N} \left(\frac{p_n^2}{2m} + \frac{\epsilon}{2} (u_{n+1} - u_n)^2 + \frac{\nu}{4} (u_{n+1} - u_n)^4 \right), \tag{1.2}$$

where p_n is the momentum of the n th oscillator, u_n is the displacement of the n th oscillator from the equilibrium position, N is the number of oscillators, ϵ is an elasticity constant, ν is the parameter of nonlinearity, and m is the mass of an oscillator. Below, the boundary conditions are chosen to be periodic, i.e., $p_{n+N} = p_n$ and $u_{n+N} = u_n$.

The system (1.2) for $\hbar = 0$ is one of the simplest models for finding conditions of appearance of stochastic properties in nonlinear systems with many degrees of freedom. It is intensively investigated since 1955, see for instance Fermi *et al.* (1955), Budincky and Bountis (1983), Berman and Izrailev (2005) and references therein. There is a certain connection between the instability of the decay type in question and the stochastic instability (Berman and Kolovskii, 1984). This latter paper presents a numerical investigation of system (1.1) in the case of initiating short waves ("narrow packet" approximation). If the parameter ν exceeds a critical value, ν_c, four wave decay processes appear corresponding to interaction of resonances (1.1). Under further increase of ν the resonances of type (1.1) strongly interact with each other, which finally results in a stochastic behavior of the chain. The availability of decay processes in a classical chain seems thus to be a preliminary step giving rise to a stochastic instability in the system.

Note in this connection that our approach based on the decay processes of the finite amplitude waves, which we analyse below, is different from the approach based on the interaction of other well-known solutions, the so-called "breathers" (Birnir *et al.*, 1994). Namely, the breathers are the solutions localised in space. Instead of this, we consider the stability of solutions in the form of the waves with finite amplitudes which provide the interaction of nonlinear resonances. We demonstrate that in spite of the quasiclassical region of the chosen parameters the quantum effects can play a significant role in the dynamics of the system.

Hence, the study of the dynamics of four wave decay processes for system (1.2) in the quantum case seems to be well motivated. This paper is organised as follows. In Sections 2, 3 and 4 we have compiled some basic facts on the dynamics of four wave decay processes in a classical chain. The equations describing the dynamics of quantum decay processes are presented in Section 5. Sections 6, 7 and 8 contain a detailed study of the quantum decay system. The local solution of the system guaranteed by the Cauchy–Kovalevskaya Theorem is proved to analytically extend to all time, space and parameter values. It is proved that in the quasiclassical limit, and under the condition of classical instability, a quantum solution converges to a classical one on the time scale

$$T \sim \frac{1}{6X} \log \frac{X}{\varepsilon},$$

where X is a dimensional action of the classical system, and ε is a quantum dimensionless parameter, $\varepsilon \sim \hbar$, which is assumed to be small in the quasiclassical region, $\varepsilon \ll 1$. We also demonstrate how to build an explicit quantum solution in the frames of the asymptotic expansions by using a small parameter, ε.

Note that the logarithmically small time-scale of applicability of a classical consideration is not related to the used mathematical approach, but is due to a physical nature of differences between classical unstable and the corresponding quantum dynamics for observables. Further analysis of this problem in relation to the experimentally realisable situations, such as Bose–Einstein condensates, quantum optical systems, nano-mechanical cantilevers and others, is of significant interest.

2. NARROW PACKET APPROXIMATION

Before discussing the decay instability in the quantum case we look more closely at some peculiarities of the dynamics of four wave decay processes in a classical chain. To this end, we pass in (1.2) to the canonical variables a_k and a_k^* by

$$a_k = \frac{1}{\sqrt{2m\hbar\omega_k}} (P_k - \imath\omega_k U_k^*), \tag{2.1}$$

where

$$P_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} p_n \, e^{-2\pi \frac{k}{N} n\imath},$$

$$U_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} u_n \, e^{2\pi \frac{k}{N} n\imath},$$

$$\omega_k = 2\sqrt{\frac{\epsilon}{m}} \, \sin \pi \, \frac{k}{N}.$$

In the classical case, the commutator $[a_j, a_k^\dagger] = 0$ vanishes for all $j, k = 1, \ldots, N$ and $I_k = \hbar \, |a_k|^2$ is a classical action of the phonon with momentum k. The Planck constant, $\hbar$, appears in the classical limit for convenience of the comparison with the quantum solution.

Suppose that the initial data of system (1.2) satisfy the condition of the "narrow packet" approximation

$$\delta k / k_0 \ll 1, \tag{2.2}$$

where $\delta k = |k - k_0|$ is the characteristic size in k of a packet of initiated modes, and k_0 is the characteristic wave number of the center of the packet ($k_0 \sim N/2$). In the variables a_k, a_k^* the Hamiltonian (1.2) takes the form

$$H = \hbar \sum_{k=1}^{N} \omega_k a_k^* a_k + \frac{1}{2} \hbar^2 \sum_{k_1, k_2, k_3, k_4} V_{k_1 k_2 k_3 k_4} \, a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4} \, \delta_{k_1 + k_2 - k_3 - k_4, 0} + R, \tag{2.3}$$

where

$$V_{k_1 k_2 k_3 k_4} = \frac{3\nu}{\epsilon m N} \left(\sin \pi \frac{k_1}{N} \, \sin \pi \frac{k_2}{N} \, \sin \pi \frac{k_3}{N} \, \sin \pi \frac{k_4}{N} \right)^{1/2}.$$

In (2.3) the terms $a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4}$ represent the resonance four wave interaction processes of waves, which are decisive under the condition (2.2). By R we denote the non-resonant terms like $a_{k_1} a_{k_2} a_{k_3} a_{k_4}$, $a_{k_1}^* a_{k_2} a_{k_3} a_{k_4}$, etc., which can be neglected under the approximation in question, at least at the initial stage.

Conditions at which the non-resonant terms can be neglected are well understood for different nonlinear systems (see for details Zakharov, 1996; Bogoliubov and Mitropolskii, 1958). In particular, in Zakharov (1996) a similar approximation was considered and explained for a classical dynamics of four wave decays in plasma. In our case, these conditions can be presented as $V_0 I / \omega_{k_0} \ll 1$, where I is the classical action (see below for notation). The latter condition is satisfied if the nonlinear part in the initial FPU Hamiltonian (1.2) is smaller than the linear one, which is assumed in our consideration.

Under the condition (2.2) one can set

$$\begin{aligned}\omega_k &\approx \omega_{k_0} + c(k - k_0) - \Omega(k - k_0)^2, \\ V_{k_1 k_2 k_3 k_4} &\approx V_0,\end{aligned} \tag{2.4}$$

where

$$c = 2\sqrt{\frac{\epsilon}{m}\frac{\pi}{N}}\cos\pi\frac{k_0}{N}, \quad \Omega = \sqrt{\frac{\epsilon}{m}}\left(\frac{\pi}{N}\right)^2 \sin\pi\frac{k_0}{N}, \quad V_0 = \frac{3\nu}{\epsilon m N}\left(\sin\pi\frac{k_0}{N}\right)^2.$$

Substituting (2.3) and (2.4) into the equations of motion

$$\imath \dot{a}_k = \frac{\partial H}{\partial a_k^*}$$

we get

$$\imath \dot{A}_j = -j^2 \Omega A_j + \hbar V_0 \sum_{j_2, j_3, j_4} A_{j_2}^* A_{j_3} A_{j_4} \delta_{j + j_2 - j_3 - j_4, 0}, \tag{2.5}$$

where

$$A_j = \exp((\omega_{k_0} + cj)t\,\imath) a_{j + k_0}.$$

3. RELATION TO THE GROSS–PITAEVSKII EQUATION

Introduce an envelope

$$\begin{aligned}A(t, \theta) &= \sum_j A_j(t) e^{\imath j \theta} \\ &= A(t, \theta + 2\pi).\end{aligned}$$

It follows from (2.5) that the function A satisfies the Gross–Pitaevskii (GP) equation (which in this case coincides with nonlinear Schrödinger equation)

$$\imath \frac{\partial A}{\partial t} = \Omega \frac{\partial^2 A}{\partial \theta^2} + \hbar V_0 |A|^2 A,$$

with periodic boundary conditions. This establishes a relation between the FPU system and the BEC system with attractive interactions (see also Berman *et al.*, 2002). One can easily see that the next order corrections in the FPU resonant Hamiltonian (2.3) lead to the modified equation for Φ (see also Berman and Izrailev, 2005)

$$\imath \frac{\partial A}{\partial t} = \Omega \frac{\partial^2 A}{\partial \theta^2} + \imath \kappa \frac{\partial^3 A}{\partial \theta^3} + \hbar V_0 |A|^2 A + 4\imath\hbar V_0 |A|^2 \frac{\partial A}{\partial \theta},$$

where $\kappa = (1/6)(\partial^3 \omega_k / \partial k^3)\,|_{k=k_0}$. This equation can be useful for understanding the peculiarities of the dynamics of the FPU system (1.2) in the narrow packet approximation, which goes beyond the integrable nonlinear Schrödinger equation. In particular, the properties of integrability of this equation and its relation to the stochastic instability in the FPU system (1.2) can be a subject of further investigations.

4. CLASSICAL PARAMETRIC INSTABILITY

The equations (2.5) describe the dynamics of four wave interactions in chain (1.2). As is shown in Berman and Kolovskii (1984), if $\nu \ll 2\pi^2 k_0/3NE_{k_0} \sim 1/E$, E being the energy of the system, then the "narrow packet" approximation survives for sufficiently large times. In what follows we think of equations (2.5) as the input ones.

As was already mentioned, the equations (2.5) are equivalent to a nonlinear Schrödinger equation with periodic boundary conditions, which is known to be a completely integrable system both in the classical and quantum cases, cf. Isergin and Korepin (1982). In spite of the integrability, the equations (2.5) describe the parametric instability, or the processes of four wave interactions (or decays).

We next present a condition for appearance of the decays. It is easy to verify that the equations (2.5) have an explicit partial solution in the form of a finite amplitude wave

$$\begin{aligned} A_k(t) &= \exp((\Omega_k - \hbar V_0 |A_k|^2) t\, \imath)\, A_k, & \\ A_j(t) &= 0, & \text{if } j \neq k, \end{aligned} \tag{4.1}$$

where $\Omega_k = k^2\Omega$. Let us examine the stability of solution (4.1) with respect to the decay of the mode k in neighbouring modes $2k \mapsto (k-l) + (k+l)$. Suppose that the modes with $j \neq k$ are slightly perturbed at the initial instant, so that $|A_j| \ll |A_k|$. By linearising equations (2.5) in A_j one easily arrives at the system

$$\begin{aligned} \imath \dot{A}_k &= -\Omega_k A_k + \hbar V_0 |A_k|^2 A_k, \\ \imath \dot{A}_{k-l} &= -\Omega_{k-l} A_{k-l} + 2\hbar V_0 |A_k|^2 A_{k-l} + \hbar V_0 A_k^2 A_{k+l}^*, \\ \imath \dot{A}_{k+l} &= -\Omega_{k+l} A_{k+l} + 2\hbar V_0 |A_k|^2 A_{k+l} + \hbar V_0 A_k^2 A_{k-l}^*. \end{aligned} \tag{4.2}$$

These equalities show that the dynamics of a "large" wave does not change at first approximation of perturbation theory. The amplitudes of "small" waves grow exponentially with the increment

$$\lambda_l = \sqrt{V_0 I(\Delta\Omega) - \left(\frac{\Delta\Omega}{2}\right)^2} = l\Omega \sqrt{\frac{2V_0 I}{\Omega} - l^2}, \tag{4.3}$$

where $\Delta\Omega = \Omega_{k-l} + \Omega_{k+l} - 2\Omega_k = 2l^2\Omega$ is a distance in frequency from the resonance (1.1), and $I = \hbar |A_k|^2$ is the classical action of the large wave. From (4.3) we get the desired condition for existence of decays, namely $2V_0 I/\Omega > 1$.

In terms of original system (1.2) this condition reads

$$\nu > \frac{\pi^2}{3NE_{k_0}} \sim \frac{1}{NE}. \tag{4.4}$$

5. QUANTUM EQUATIONS OF DECAY

We now pass to analysis of the quantum case where p_n and u_n in (1.2) are operators with commutativity relation $[u_j, p_k] = \imath\hbar\,\delta_{jk}$. Changing the variables by (2.1) and (2.4) as in the classical case (see Berman *et al.*, 1986, for more details), we get the following system of operator equations which describe the four wave interactions in the quantum case

$$\imath \dot{A}_j = -j^2(1+q)\Omega\, A_j + \hbar V_0 \sum_{j_2, j_3, j_4} A^{\dagger}_{j_2} A_{j_3} A_{j_4}\, \delta_{j+j_2-j_3-j_4,0}, \tag{5.1}$$

where

$$[A_j, A_k^{\dagger}] = \delta_{jk},$$

$$q = \hbar\,\frac{\nu \cot\dfrac{\pi}{2N}}{32N\sqrt{m\epsilon^3}},$$

Ω and V_0 being defined in (2.4). The renormalisation of the frequency Ω is due to the ordering of operators. It will cause no confusion if we use the same notation Ω_j to designate $j^2(1+q)\Omega$.

To treat the system (5.1) we use the techniques of projection onto the basis of coherent states, cf. Berman *et al.* (1981) and Sinitsyn and Tsukernik (1982). Assume that at the initial instant each mode of the bosonic field rests on a coherent state described by a number α_j. We denote

$$\alpha_j(t) = \langle\vec{\alpha}|\, A_j(t)\, |\vec{\alpha}\rangle$$
$$= \alpha_j(t, \vec{\alpha}, \vec{\alpha}^*),$$

where $|\vec{\alpha}\rangle$ is the vector of coherent states of the phononic field at the initial instant. The coherent states are chosen as initial wave functions for phonons because these states provide the most close description of the quantum dynamics to the classical one. Our consideration is based on deriving a closed system of equations for quantum observables. To do this, we use a Heisenberg representation in which the operators depend on time, and the wave function is time-independent and is represented as a product of coherent states for individual phonons.

From (5.1) it follows that the operator $A_j(t)$ satisfies the Heisenberg equation

$$\imath\hbar \dot{A}_j = [A_j(t), H_{\text{eff}}], \tag{5.2}$$

with the effective Hamiltonian

$$H_{\text{eff}} = -\hbar \sum_k \Omega_k A_k^\dagger A_k + \frac{1}{2}\hbar^2 V_0 \sum_{k_1,k_2,k_3,k_4} A_{k_1}^\dagger A_{k_2}^\dagger A_{k_3} A_{k_4}\, \delta_{k_1+k_2-k_3-k_4,0}.$$

Averaging the equation (5.2) in the initial coherent state gives a closed partial differential equation for quantum observable $\alpha(t, \vec{\alpha}, \vec{\alpha}^*)$

$$\begin{aligned} \imath \dot{\alpha}_j(t) &= \hat{T}\, \alpha_j(t), \\ \alpha_j(0) &= \alpha_j, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} \hat{T} = &-\sum_k \Omega_k \left(\alpha_k \frac{\partial}{\partial \alpha_k} - \text{c.c.}\right) \\ &+\hbar V_0 \sum_{k_1,k_2,k_3,k_4} \left(\alpha_{k_1}^* \alpha_{k_2} \alpha_{k_3} \frac{\partial}{\partial \alpha_{k_4}} - \text{c.c.}\right) \delta_{k_1+k_2-k_3-k_4,0} \\ &+\frac{1}{2}\hbar V_0 \sum_{k_1,k_2,k_3,k_4} \left(\alpha_{k_1} \alpha_{k_2} \frac{\partial}{\partial \alpha_{k_3}} \frac{\partial}{\partial \alpha_{k_4}} - \text{c.c.}\right) \delta_{k_1+k_2-k_3-k_4,0}, \end{aligned}$$

the c.c. meaning complex conjugate terms (cf. Berman *et al.*, 1986, for more details).

The Equation (5.3) is easily checked to possess a solution of the form of finite amplitude periodic wave

$$\begin{aligned} \alpha_k(t) &= \exp(\Omega_k t\imath - (1 - \exp(-\hbar V_0 t\imath))|\alpha_k|^2)\, \alpha_k, \\ \alpha_j(t) &= 0, \qquad\qquad \text{if } j \neq k. \end{aligned} \tag{5.4}$$

Note that the solution (5.4) turns into the classical wave (4.1) when $\hbar \to 0$, $|\alpha_k| \to \infty$, and $\hbar|\alpha_k|^2 \to I$. We now examine the stability of solution (5.4) relative to the decay in neighbouring modes $2k \mapsto (k - l) + (k + l)$. Assume that at the initial instant the amplitudes of the modes $j \neq k$ are small, i.e., $|\alpha_j| \ll |\alpha_k|$. In this case one can look for a solution α_{k+l} of (5.4) in the form of expansion in α_j,

$$\begin{aligned} \alpha_{k+l}(t, \vec{\alpha}, \vec{\alpha}^*) = &\; c_{l,0}(t, \alpha_k, \alpha_k^*) \\ &+ \sum_{j \neq 0} \left(c_{l,j}^{(1,0)}(t, \alpha_k, \alpha_k^*)\alpha_{k+j} + c_{l,j}^{(0,1)}(t, \alpha_k, \alpha_k^*)\alpha_{k+j}^*\right) \\ &+ \ldots, \end{aligned} \tag{5.5}$$

the dots meaning the terms containing the products $\alpha_{k+j_1}\alpha_{k+j_2}$, $\alpha_{k+j_1}^*\alpha_{k+j_2}$, $\alpha_{k+j_1}^*\alpha_{k+j_2}^*$, etc. From the initial condition $\alpha_{k+l}(0, \vec{\alpha}, \vec{\alpha}^*) = \alpha_{k+l}$ we readily deduce that

$$\begin{aligned} c_{0,0}(0, \alpha_k, \alpha_k^*) &= \alpha_k, \quad c_{l,0}(0, \alpha_k, \alpha_k^*) = 0; \\ c_{l,j}^{(1,0)}(0, \alpha_k, \alpha_k^*) &= \delta_{lj}, \quad c_{l,j}^{(0,1)}(0, \alpha_k, \alpha_k^*) = 0 \end{aligned} \tag{5.6}$$

for $l \neq 0$. In (5.5) α_{k+j} and α_{k+j}^* are the initial amplitudes of "small" waves, and α_k is the initial amplitude of a "large" wave. The coefficients $c_{l,0}$, $c_{l,j}^{(1,0)}$ and $c_{l,j}^{(0,1)}$, etc. do not explicitly contain smallness related to the amplitudes α_{k+j} with $j \neq 0$.

Below, we will study the dynamics of functions $c_{l,0}$, $c_{l,j}^{(1,0)}$ and $c_{l,j}^{(0,1)}$, for they determine the evolution of small perturbations with amplitudes α_{k+j}. Substituting (5.5) into (5.3) and gatherings the coefficients of the same powers of α_{k+j}, we arrive at a system of equations for the coefficients which is not closed in general, i.e., the equations for $c_{l,0}$, $c_{l,j}^{(1,0)}$ and $c_{l,j}^{(0,1)}$ also include higher order coefficients. However, one can show that higher order coefficients describe the influence of small waves on each other and on the large wave. Hence they do not essentially contribute to the dynamics of the system at the initial stage. A quasiclassical asymptotics of the contribution of higher order coefficients is discussed in Berman *et al.* (1986).

On account of the above remark, we cut off the expansion (5.5) upon the linear terms. In this way we get the following closed system of differential equations

$$\begin{aligned} \imath\, \dot{c}_{l,0} &= \hat{M} c_{l,0}, \\ \imath\, \dot{c}_{l,j}^{(1,0)} &= \hat{M} c_{l,j}^{(1,0)} - (\Omega_{k+j} - 2\hbar V_0 |\alpha_k|^2) c_{l,j}^{(1,0)} + 2\hbar V_0 \alpha_k \frac{\partial}{\partial \alpha_k} c_{l,j}^{(1,0)} - \hbar V_0 \alpha_k^{*2} c_{l,-j}^{(0,1)}, \\ \imath\, \dot{c}_{l,-j}^{(0,1)} &= \hat{M} c_{l,-j}^{(0,1)} - (\Omega_{k-j} - 2\hbar V_0 |\alpha_k|^2) c_{l,-j}^{(0,1)} - 2\hbar V_0 \alpha_k \frac{\partial}{\partial \alpha_k} c_{l,-j}^{(0,1)} + \hbar V_0 \alpha_k^2 c_{l,j}^{(1,0)}, \end{aligned} \tag{5.7}$$

cf. Berman *et al.* (1986), where

$$\hat{M} = -(\Omega_k - \hbar V_0 |\alpha_k|^2)\, \alpha_k \frac{\partial}{\partial \alpha_k} + \frac{1}{2} \hbar V_0\, \alpha_k^2 \frac{\partial^2}{\partial \alpha_k^2} - \text{c.c.},$$

the c.c. stand for complex conjugate terms.

The solution of the first Equation (5.7) has the form (5.4) and describes the dynamics of a "large" wave at first approximation. The remaining system of two equations can be further simplified. For this purpose we conclude from (5.6) and the linearity of (5.7) that only two relevant summands in (5.5) are different from zero, namely $c_{l,l}^{(1,0)}$ and $c_{l,-l}^{(0,1)}$. Let us now substitute the unknown functions by

$$\begin{aligned} c_{l,l}^{(1,0)} &= \exp(-(\Omega_{k-l} - 2\Omega_k)t\imath) f, \\ c_{l,-l}^{(0,1)} &= \frac{\alpha_k}{\alpha_k^*} \exp(-(\Omega_{k-l} - 2\Omega_k)t\imath) g. \end{aligned} \tag{5.8}$$

Under this notation the average of the operator $A_{k+l}(t)$ is

$$\alpha_{k+l}(t) = \langle\vec{\alpha}|\, A_{k+l}(t)\, |\vec{\alpha}\rangle$$
$$= \exp(-(\Omega_{k-l} - 2\Omega_k)t\imath)(\alpha_{k+l} f(t) + \frac{\alpha_k}{\alpha_k^*}\alpha_{k-l}^* g(t)).$$

Substituting (5.8) into (5.7) we deduce that both f and g depend only on $|\alpha_k|^2$ and satisfy the system of equations

$$\begin{aligned} \imath\, \dot{f} &= (2V_0 I - (\Omega_{k-l} + \Omega_{k+l} - 2\Omega_k))f + 2\hbar V_0\, I \frac{\partial f}{\partial I} - V_0 I g, \\ \imath\, \dot{g} &= V_0 I\, f + \hbar V_0\, g \end{aligned} \tag{5.9}$$

with initial data

$$\begin{aligned} f(0) &= 1, \\ g(0) &= 0, \end{aligned} \tag{5.10}$$

where $I = \hbar|\alpha_k|^2$ stands, as before, for the classical action of the kth mode.

Equation (5.9) describes the decay instability in the quantum case. From now on they will be referred to as equations of quantum decay. In the classical case $\hbar = 0$ they can be solved explicitly, which shows once again the exponential growth of "small" waves with the increment λ_l (4.3) provided that $2V_0 I > \Omega$, cf. Section 2.

6. ANALYSIS OF QUANTUM EQUATIONS

Before we pass to the analysis of Equations (5.9) we make necessary simplifications. We reset

$$\begin{aligned} f &\mapsto \exp(-\hbar V_0 t\, \imath) f, \\ g &\mapsto \exp(-\hbar V_0 t\, \imath) g \end{aligned}$$

and assume, for simplicity, $2(1+q)\Omega - \hbar V_0 \approx 2\Omega$. We also introduce a dimensionless time $\Omega t \mapsto t$ and a dimensionless variable $x = V_0 I/\Omega$ instead of action, I. For simplicity we restrict our attention to the case $l = 1$. Then (5.9) takes the form

$$\begin{aligned} \imath\, \dot{f} &= 2(x-1)f + 2\varepsilon x \frac{\partial f}{\partial x} - xg, \\ \imath\, \dot{g} &= xf, \end{aligned} \tag{6.1}$$

where

$$\varepsilon = \hbar \frac{V_0}{\Omega}$$

is a dimensionless quantum parameter.

The system (6.1) is of mixed type with hyperbolic degeneracy on the line $x = 0$. The general theory yields merely that (6.1) has a real analytic solution in (t, x, ε) in some neighborhood of the plane $t = 0$. We prove in Section 7. that this solution actually extends analytically in (t, x, ε) to all of $\mathbb{R}^3$, the extension satisfying

$$\sqrt{|f|^2 + |g|^2} \le \frac{\sqrt{5}}{2} \exp\left(\frac{5}{2} xt\right) \tag{6.2}$$

for all t, x and ε.

The last inequality shows that decays in the quantum case run not faster than $\exp(\gamma t)$, where γ does not depend on t. This enable us to apply the Laplace transform in the analysis of system (6.1).

Since the solution of (6.1) for $\varepsilon = 0$ has an explicit analytic form, it is interesting to develop a quasiclassical approach for describing the dynamics of decays. Denote by $f_{\rm cl}$, $g_{\rm cl}$ the solution of (6.1) for $\varepsilon = 0$. We prove in Section 8. that

$$f(t, x, \varepsilon) = \sum_{k=0}^{\infty} \Psi^k f_{\rm cl}(t, x)\, \varepsilon^k, \tag{6.3}$$

Ψ being the integro-differential operator

$$\Psi u\,(t, x) = -2\imath x \int_0^t f_{\rm cl}(t - s, x)\, \frac{\partial u}{\partial x}(s, x)\, ds.$$

The series (6.3) converges uniformly on all subsets of $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ of the form

$$\{t \le T\} \times \{x \le X\} \times \{\varepsilon \le (2Te^{3XT})^{-1}\}.$$

Hence it follows that

$$T \sim \frac{1}{6X} \log \frac{X}{\varepsilon}$$

in the domain of quasiclassical approach $x/\varepsilon = I/\hbar \gg 1$. The time of applicability of the quasiclassical approach is therefore logarithmically small, i.e., $T \sim \log 1/\hbar$ in contrast to $T \sim 1/\hbar^{\gamma}$ for classically stable dynamics. This is a consequence of instability of the dynamics of the classical system.

A similar result was earlier obtained in Berman and Zaslavskiis (1982) when studying conditions of applicability of quasiclassical approximations for describing dynamics of nonlinear quantum system whose classical limit has the property of stochastic instability.

7. EXISTENCE OF SOLUTIONS

Let us formulate the problem more precisely. By (6.1), we have the following system for approximate description of the dynamics of quantum decays

$$\begin{aligned} \dot{f} &= -2\imath(x-1)f - 2\imath\varepsilon x\frac{\partial f}{\partial x} + \imath x g, \\ \dot{g} &= -\imath x f \end{aligned} \tag{7.1}$$

in the half-plane $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ under the initial conditions

$$\begin{aligned} f(0, x) &= 1, \\ g(0, x) &= 0. \end{aligned} \tag{7.2}$$

In fact, the domain of $x = V_0 I/\Omega$ is $x > 1/2$, the last condition guarantying the existence of decays by (4.4).

The principal symbol of (7.1) is given by the matrix

$$\begin{pmatrix} \imath\tau & -2\varepsilon x\xi & 0 \\ & 0 & \imath\tau \end{pmatrix}$$

with the determinant $-\tau(\tau + 2\imath\,\varepsilon x\xi)$. It follows that (7.1) is a mixed type system with hyperbolic degeneracy on the line $x = 0$. The real characteristics of this system are lines $x = \text{const}$, hence the Cauchy problem (7.1), (7.2) is noncharacteristic.

The system (7.1) has normal form with respect to the time variable t, and the coefficients of the system and the Cauchy data (7.2) are entire functions of t, x and ε. Therefore, it fulfills the conditions of the Cauchy–Kovalevskaya Theorem, which implies that the problem (7.1), (7.2) has a real analytic solution

$$F(t, x, \varepsilon) = \begin{pmatrix} f(t, x, \varepsilon) \\ g(t, x, \varepsilon) \end{pmatrix}$$

in some neighbourhood U of the hyperplane $\{t = 0\}$ in $\mathbb{R}^3$. The solution is unique in the class of real analytic functions. Moreover, the solution is unique in the class of continuously differentiable functions, which is due to Holmgren's uniqueness theorem.

The question arises whether the solution actually extends analytically to all of the half-space $\{t > 0\}$. To treat the problem we eliminate one unknown function of the system.

Lemma 7.1. *Given any entire function $\Phi_0(x)$, the Cauchy problem for the truncated equation*

$$\begin{cases} \dot{\Phi} = -2\imath(x-1)\Phi - 2\imath\varepsilon x\dfrac{\partial\Phi}{\partial x} & \textit{for } t > 0, \\ \Phi(0) = \Phi_0(x) \end{cases} \tag{7.3}$$

has a unique solution which is an entire function of (t, x, ε).

Proof. Since the functions we work with are entire we can change the variables by

$$\begin{aligned} t &= \imath z, \\ \log x &= -2\varepsilon\, z + w, \end{aligned}$$

with $z, w \in \mathbb{R}$. For the function $u(z, w) := \Phi(\imath z, \exp(-2\varepsilon\, z + w))$ the Cauchy problem (7.3) becomes

$$\begin{cases} \dfrac{\partial u}{\partial z} &= 2(e^{-2\varepsilon z + w} - 1)u \quad \text{for} \quad z \in \mathbb{R}, \\ u(0, w) = \Phi_0(e^w). \end{cases}$$

This latter problem has a unique entire solution which can be moreover explicitly written,

$$u(z, w) = \Phi_0(e^w)\, \exp\left(\frac{1}{\varepsilon} e^w (1 - e^{-2\varepsilon z}) - 2z\right).$$

Returning to the variables t and x yields

$$\Phi(t, x, \varepsilon) = \Phi_0(x)\, \exp\left(\frac{x}{\varepsilon}(e^{-2\imath\, \varepsilon t} - 1) + 2\imath\, t\right), \tag{7.4}$$

as desired.

□

Note that $\Phi(t, x, \varepsilon)$ converges to $\Phi_0(x)\, \exp(-2\imath\, t(x - 1))$ for $\varepsilon \to 0$, as is easy to see.

From now on we tacitly assume that $\Phi_0 = 1$. By abuse of notation, we use the same letter Φ to designate the solution of (7.3) with $\Phi_0 = 1$. Set

$$\Gamma(t, x, \varepsilon) = -\imath x \int_0^t \Phi(s, x, \varepsilon)\, ds. \tag{7.5}$$

Lemma 7.2. *Suppose* P *is a continuous function of* (t, x, ε) *in the half-space* $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$. *Then the solution of the Cauchy problem for the system*

$$\begin{aligned} \dot{f} &= -2\imath(x - 1)f - 2\imath\varepsilon x \frac{\partial f}{\partial x} + P, \\ \dot{g} &= -\imath x f \end{aligned} \tag{7.6}$$

under initial conditions (7.2) is given by the formula

$$f(t, x, \varepsilon) = \Phi(t, x, \varepsilon) + \int_0^t \Phi(t - s, x, \varepsilon) P(s, x, \varepsilon)\, ds,$$

$$g(t, x, \varepsilon) = \Gamma(t, x, \varepsilon) + \int_0^t \Gamma(t - s, x, \varepsilon) P(s, x, \varepsilon)\, ds.$$

Proof. To simplify notation, we will not indicate the dependence of f, g, etc. on x and ε.

Since $\Phi(0) = 1$ and $\Gamma(0) = 0$ both f and g satisfy (7.2). Furthermore, an easy calculation shows that

$$\begin{aligned}
\dot{f}(t) &= \dot{\Phi}(t) + \Phi(0)P(t) + \int_0^t \dot{\Phi}(t-s)P(s)\,ds \\
&= \dot{\Phi}(t) + \Phi(0)P(t) - \int_0^t \left(2\imath(x-1)\Phi(t-s) + 2\imath\varepsilon x \frac{\partial}{\partial x}\Phi(t-s)\right) P(s)\,ds \\
&= \dot{\Phi}(t) + \Phi(0)P(t) - 2\imath(x-1)(f(t)-\Phi(t)) - 2\imath\varepsilon x \frac{\partial}{\partial x}(f(t)-\Phi(t)) \\
&= -2\imath(x-1)f(t) - 2\imath\varepsilon x \frac{\partial}{\partial x} f(t) + P(t),
\end{aligned}$$

the last equality being a consequence of (7.3), and similarly

$$\begin{aligned}
\dot{g}(t) &= \dot{\Gamma}(t) + \Gamma(0)P(t) + \int_0^t \dot{\Gamma}(t-s)P(s)\,ds \\
&= \dot{\Gamma}(t) + \Gamma(0)P(t) - \imath x \int_0^t \Phi(t-s)P(s)\,ds \\
&= \dot{\Gamma}(t) + \Gamma(0)P(t) - \imath x\,(f(t)-\Phi(t)),
\end{aligned}$$

showing the lemma.

□

Lemma 7.2 allows one to reduce the Cauchy problem (7.1), (7.2) to an integral equation of Volterra type, namely

$$\begin{aligned}
f(t) &= \Phi(t) + \imath x \int_0^t \Phi(t-s)g(s)\,ds, \\
g(t) &= \Gamma(t) + \imath x \int_0^t \Gamma(t-s)g(s)\,ds.
\end{aligned} \tag{7.7}$$

Theorem 7.3. *The problem (7.1), (7.2) has a unique solution* $\{f, g\}$ *which is a real analytic function of* (t, x, ε) *on all of* $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$*, satisfying (6.2).*

Proof. Since both Φ and Γ are entire functions of (t, x, ε), the existence and uniqueness of a solution follow from the classical Volterra theory. This solution can be actually obtained by successive approximations. It remains to establish (6.2).

To this end, we apply the successive approximation method to solve the second equation of (7.7), and then we substitute g to the first equation, thus obtaining f. For simplicity, we restrict our discussion to the case of nonnegative t,

x and ε, which involves no loss of generality. Setting $g_0 = \Gamma$, we define iterations

$$g_k(t) = \Gamma(t) + \imath x \int_0^t \Gamma(t-s) g_{k-1}(s)\, ds$$

for $k = 1, 2, \ldots$.

Since

$$\begin{aligned} |g_0(t)| &\le x \int_0^t \exp\left(\frac{x}{\varepsilon}(\cos 2\varepsilon s - 1)\right) ds \\ &\le x \int_0^t \exp\left(\frac{x}{\varepsilon}\, 2\varepsilon s\right) ds \\ &\le \frac{1}{2}(e^{2tx} - 1), \end{aligned}$$

one easily obtains by induction

$$\begin{aligned} |g_1(t)| &\le \left(\frac{1}{2} - \frac{1}{4}\right)\varphi + \frac{1}{4}\, tx\, \psi, \\ |g_2(t)| &\le \left(\frac{1}{2} - \frac{1}{16} + \frac{1}{16}(tx)^2\right)\varphi + \frac{1}{16}\, tx\, \psi, \\ |g_3(t)| &\le \left(\frac{1}{2} - \frac{21}{96}\right)\varphi + \left(\frac{21}{96}\, tx + \frac{1}{96}(tx)^3\right)\psi, \\ |g_4(t)| &\le \left(\frac{1}{2} - \frac{63}{768} + \frac{45}{768}(tx)^2 + \frac{1}{768}(tx)^4\right)\varphi + \left(\frac{63}{768}\, tx - \frac{2}{768}(tx)^3\right)\psi, \end{aligned}$$

where

$$\begin{aligned} \varphi &= e^{2tx} - 1, \\ \psi &= e^{2tx} + 1. \end{aligned}$$

Given any $k = 1, 2, \ldots$, we get

$$|g_k(t)| \le (c_{k,0} + c_{k,2}(tx)^2 + \ldots)\varphi + (c_{k,1}tx + c_{k,3}(tx)^3 + \ldots)\psi,$$

where $c_{k,n} = 0$ for $n > k$. The coefficients $c_{k,n}$ can actually be estimated uniformly in k by

$$|c_{k,n}| \le \frac{1}{2}\,\frac{1}{2^n}\,\frac{1}{n!} \tag{7.8}$$

for all n.

Letting $k \to \infty$ we deduce that the limiting function $g(t)$ fulfills the estimate

$$\begin{aligned} |g(t)| &\le \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{tx}{2}\right)^{2n} \varphi + \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{tx}{2}\right)^{2n+1} \psi \\ &\le \frac{1}{2}\cosh\frac{tx}{2}\,(e^{2tx} - 1) + \frac{1}{2}\sinh\frac{tx}{2}\,(e^{2tx} + 1) \end{aligned}$$

for all $t \geq 0$. Using the definitions of functions $\cosh x$ and $\sinh x$ we readily obtain

$$|g(t)| \leq \frac{1}{2}\left(e^{(5/2)\,tx} - e^{-(1/2)\,tx}\right) \leq \frac{1}{2}\,e^{(5/2)\,tx}$$

and

$$|f(t)| \leq e^{2tx} + x\int_0^t e^{2x(t-s)}\,\frac{1}{2}\,e^{(5/2)\,sx}\,ds \leq e^{(5/2)\,tx}, \qquad (7.9)$$

which implies (6.2).

□

As already mentioned, the solution $\{f, g\}$ of (7.1), (7.2) is also unique in the space of continuously differentiable functions.

8. SUCCESSIVE APPROXIMATIONS

Set

$$A = \begin{pmatrix} -2\imath\,(x-1) & \imath x \\ -\imath x & 0 \end{pmatrix},$$

and let

$$\lambda_+ = \imath(1-x) + \sqrt{2x-1},$$
$$\lambda_- = \imath(1-x) - \sqrt{2x-1}$$

stand for the eigenvalues of the matrix A. The system (7.1) for $\varepsilon = 0$ takes the form

$$\dot{F}_{\mathrm{cl}} = A F_{\mathrm{cl}}$$

with

$$F_{\mathrm{cl}}(t,x) = \begin{pmatrix} f_{\mathrm{cl}}(t,x) \\ g_{\mathrm{cl}}(t,x) \end{pmatrix},$$

hence the solution of the Cauchy problem (7.1), (7.2) corresponding to $\varepsilon = 0$ can be written in the form

$$F_{\mathrm{cl}} = \sum_{k=0}^{\infty} A^k F_0(x)\,\frac{t^k}{k!}, = \begin{pmatrix} e^{\lambda_- t} + \lambda_+\,\dfrac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \\ -\imath\,\dfrac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \end{pmatrix} \qquad (8.1)$$

where

$$F_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The inequality (7.9) certainly applies to f_{cl}, thus giving an estimate for all real t and x. In order to derive an estimate of F_{cl} on all of $\mathbb{C} \times \mathbb{C}$ we need the following lemma.

Lemma 8.1. *As defined above, F_{cl} is an entire function of t and x, satisfying*

$$|F_{\mathrm{cl}}(t, x)| \leq \exp\left((2|1 - x| + |x|)|t|\right)$$

for all $(t, x) \in \mathbb{C} \times \mathbb{C}$.

Proof. To shorten notation, set $z = 2\imath\,(1 - x)$ and $w = \imath x$. An easy calculation shows that

$$A^k F_0\,(x) = \begin{pmatrix} c_k z^k - c_{k-2} z^{k-2} w^2 + c_{k-4} z^{k-4} w^4 - \dots \\ -c_{k-1} z^{k-1} w + c_{k-3} z^{k-3} w^3 + c_{k-5} z^{k-5} w^5 + \dots \end{pmatrix},$$

where the coefficients $c_k, c_{k-1}, \dots, c_0$ are natural numbers determined by the table

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 1 \\
1 \quad 1 \quad 2 \quad 1 \\
1 \quad 1 \quad 3 \quad 2 \quad 1 \\
1 \quad 1 \quad 4 \quad 3 \quad 3 \quad 1 \\
1 \quad 1 \quad 5 \quad 4 \quad 6 \quad 3 \quad 1 \\
1 \quad 1 \quad 6 \quad 5 \quad 10 \quad 6 \quad 4 \quad 1 \\
1 \quad 1 \quad 7 \quad 6 \quad 15 \quad 10 \quad 10 \quad 4 \quad 1 \\
1 \quad 1 \quad 8 \quad 7 \quad 21 \quad 15 \quad 20 \quad 10 \quad 5 \quad 1 \\
1 \quad 1 \quad 9 \quad 8 \quad 28 \quad 21 \quad 35 \quad 20 \quad 15 \quad 5 \quad 1 \\
c_k \quad c_{k-1} \qquad \dots \qquad c_1 \quad c_0.
\end{array} \tag{8.2}$$

Using the inequality $\sqrt{|a|^2 + |b|^2} \leq |a| + |b|$ and comparing (8.2) with the Pascal triangle we get

$$\begin{aligned} |A^k F_0\,(x)| &\leq |z|^k + (|z| + |w|)^{k-1}|w| \\ &\leq (|z| + |w|)^k. \end{aligned}$$

Substituting this estimate into (8.1) yields

$$\begin{aligned} |F_{\mathrm{cl}}(t, x)| &\leq \sum_{k=0}^{\infty} |A^k F_0\,(x)|\,\frac{|t|^k}{k!}, \\ &\leq \exp\left((|z| + |w|)|t|\right) \end{aligned}$$

for all $(t, x) \in \mathbb{C} \times \mathbb{C}$, as desired.

□

The proof can be summarised by saying that the matrix A has an operator bound less that the Holmgren norm $2|1 - x| + |x|$ from which the estimate is immediate.

Theorem 7.3 shows immediately that $F(t, x, 0) = F_{\mathrm{cl}}(t, x)$ for all t and x, i.e., the classical solution is the pointwise limit of the quantum solution if $\varepsilon \to 0$. Given any small $\varepsilon > 0$, the question arises of the range of times t for which the classical limit still satisfactory describes the dynamics of quantum decays.

To study the problem we make use of the geometric series to get an asymptotic expansion of $F(t, x, \varepsilon)$ in powers of ε.

Lemma 8.2. *Let P be a continuous function of (t, x) in the quarter-plane $\mathbb{R}_+ \times \mathbb{R}_+$. Then the solution of the Cauchy problem for the system*

$$\begin{aligned} \dot{f} &= -2\imath(x-1)f + \imath x g + P, \\ \dot{g} &= -\imath x f \end{aligned} \tag{8.3}$$

under initial conditions (7.2) is given by the formula

$$f(t, x) = f_{\mathrm{cl}}(t, x) + \int_0^t f_{\mathrm{cl}}(t-s, x)P(s, x)\, ds,$$

$$g(t, x) = g_{\mathrm{cl}}(t, x) + \int_0^t g_{\mathrm{cl}}(t-s, x)P(s, x)\, ds.$$

These formulas are just the well-known Duhamel formulas for an inhomogeneous linear evolution system.

Proof. To shorten notation, we write $f(t)$ and $g(t)$ instead of $f(t, x)$, $g(t, x)$, etc.

Since $f_{\mathrm{cl}}(0) = 1$ and $g_{\mathrm{cl}}(0) = 0$ both f and g satisfy (7.2). Furthermore, an easy calculation shows that

$$\begin{aligned} \dot{f}(t) &= \dot{f}_{\mathrm{cl}}(t) + f_{\mathrm{cl}}(0)P(t) + \int_0^t \dot{f}_{\mathrm{cl}}(t-s)P(s)\, ds \\ &= \dot{f}_{\mathrm{cl}}(t) + f_{\mathrm{cl}}(0)P(t) + \int_0^t \left(-2\imath(x-1)f_{\mathrm{cl}}(t-s) + \imath x g_{\mathrm{cl}}(t-s)\right) P(s)\, ds \\ &= \dot{f}_{\mathrm{cl}}(t) + f_{\mathrm{cl}}(0)P(t) - 2\imath(x-1)(f(t) - f_{\mathrm{cl}}(t)) + \imath x(g(t) - g_{\mathrm{cl}}(t)) \\ &= -2\imath(x-1)f(t) + \imath x g(t) + P(t), \end{aligned}$$

and similarly

$$\begin{aligned}\dot{g}(t) &= \dot{g}_{\rm cl}(t) + g_{\rm cl}(0)P(t) + \int_0^t \dot{g}_{\rm cl}(t-s)P(s)\,ds \\ &= \dot{g}_{\rm cl}(t) + g_{\rm cl}(0)P(t) - \imath x \int_0^t f_{\rm cl}(t-s)P(s)\,ds \\ &= \dot{g}_{\rm cl}(t) + g_{\rm cl}(0)P(t) - \imath x\,(f(t) - f_{\rm cl}(t)),\end{aligned}$$

which completes the proof.

□

Using Lemma 7.2 reduces the Cauchy problem (7.1), (7.2) to an integral equation of Volterra type, namely

$$f(t,x,\varepsilon) = f_{\rm cl}(t,x) - 2\imath\varepsilon x \int_0^t f_{\rm cl}(t-s,x)\frac{\partial f}{\partial x}(s,x,\varepsilon)\,ds. \tag{8.4}$$

Equation (8.4) is an in general difficult integral equation to analyse, not lending itself to for example analysis in a Sobolev space, for the operator does not contract. Usually these linear equations are analysed including the $x\partial_x$ as part of the unperturbed operator (via characteristics). It is important to note that the characteristics go to infinity in an infinite time, from which the global existence follows.

As in Section 6, we denote by Ψ the integro-differential operator

$$\Psi u\,(t,x) = -2\imath x \int_0^t f_{\rm cl}(t-s,x)\,\frac{\partial u}{\partial x}(s,x)\,ds,$$

then the equation (8.4) can be written in the form

$$(I - \varepsilon\Psi)f = f_{\rm cl}$$

whence

$$\begin{aligned}f(t,x,\varepsilon) &= (I-\varepsilon\Psi)^{-1} f_{\rm cl}\,(t,x) \\ &= \sum_{k=0}^{\infty} \Psi^k f_{\rm cl}\,(t,x)\,\varepsilon^k.\end{aligned} \tag{8.5}$$

One verifies by induction that

$$\Psi^k f_{\rm cl}\,(t,x) = x\left(\left(\sum_{j=0}^{k} c_{-,k,j}(x)t^j\right)e^{\lambda_- t} + \left(\sum_{j=0}^{k} c_{+,k,j}(x)t^j\right)e^{\lambda_+ t}\right)$$

for $k = 1, 2, \ldots$, where $c_{\pm,k,j}(x)$ are irrational functions having the only singularity at the point $x = 1/2$. Since $f_{\rm cl}$ is an entire function, the iterations $\Psi^k f_{\rm cl}$ are entire functions of t and x, too.

Note that (8.5) is a regular asymptotic series in powers of the small parameter. No boundary layer is required, for the degeneracy at $\varepsilon = 0$ does not affect the nature of the Cauchy problem.

Theorem 8.3. *The series (8.5) converges uniformly in t, x and ε on compact subsets of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ of the form*

$$\{|t| \le T\} \times \{|x| \le X\} \times \{|\varepsilon| \le (2Te^{3XT})^{-1}\}.$$

Proof. From the Cauchy formula it follows that if $\varphi(x)$ is an entire function of $x \in \mathbb{R}$ then

$$\sup_{|z| \le r'X} \left| \frac{\partial \varphi}{\partial z} \right| \le \frac{1}{(r - r')X} \sup_{|z| \le rX} |\varphi(z)| \tag{8.6}$$

for all $X > 0$ and $0 < r' < r$.

By Lemma 8.1 we conclude that

$$\sup_{|z| \le rX} |f_{\mathrm{cl}}(t, z)| \le e^{(3rX+2)|t|}$$

for any $r > 0$. We next show by induction that for all $k = 1, 2, \ldots$ the estimate holds

$$\sup_{|z| \le X/k+1} |\Psi^k f_{\mathrm{cl}}(t, z)| \le (2|t|)^k \, e^{(3X+2)|t|}. \tag{8.7}$$

For $k = 1$ we get, by (8.6),

$$\begin{aligned}
\sup_{|z| \le X/2} |\Psi f_{\mathrm{cl}}(t, z)| &\le \sup_{|z| \le X/2} |-2\imath z| \int_0^t |f_{\mathrm{cl}}(t-s, z)| \, |(\partial/\partial z) f_{\mathrm{cl}}(s, z)| \, ds \\
&\le X \int_0^t \exp\left(\left(3\frac{X}{2} + 2 \right)(t-s) \right) \frac{2}{X} \exp((3X+2)s) \, ds \\
&= 2 \exp\left(\left(3\frac{X}{2} + 2 \right) t \right) \int_0^t \exp\left(3\frac{X}{2} s \right) ds \\
&\le 2|t| \exp((3X+2)|t|),
\end{aligned}$$

as desired. Having granted the inequalities (8.7) up to the number k, we derive, by (8.6),

$$\begin{aligned}
&\sup_{|z| \le X/k+2} |\Psi^{k+1} f_{\mathrm{cl}}(t, z)| \\
&\le \frac{2X}{k+2} \int_0^t \exp\left(\left(3\frac{X}{k+2} + 2 \right)(t-s) \right) \frac{(k+1)(k+2)}{X} \sup_{|z| \le X/k+1} |\Psi^k f_{\mathrm{cl}}(s, z)| ds
\end{aligned}$$

$$\leq 2(k+1)\,\exp\left(\left(3\frac{X}{k+2}+2\right)t\right)\int_0^t (2s)^k \exp\left(3\,\frac{k+1}{k+2}Xs\right)ds$$

$$\leq 2(k+1)\,\exp\left((3X+2)|t|\right)\int_0^t (2s)^k\,ds$$

$$\leq (2|t|)^{k+1}\,\exp\left((3X+2)|t|\right),$$

thus completing the induction step.

Since X is actually arbitrary in (8.7) we easily deduce from this inequality that

$$\sup_{|z|\leq X}|\Psi^k f_{\rm cl}(t,z)| \leq e^{(3X+2)|t|}\left(2|t|e^{3X|t|}\right)^k$$

for all $t \in \mathbb{R}$. Hence it follows that the series (8.5) converges uniformly in t, x and ε on each compact set

$$\{|t|\leq T\}\times\{|x|\leq X\}\times\{|\varepsilon|\leq (2Te^{3XT})^{-1}\},$$

for

$$|f(t,x,\varepsilon)| \leq \exp\left((3X+2)|t|\right)\sum_{k=0}^{\infty}(2|\varepsilon||t|\,\exp(3X|t|))^k$$

$$\leq \frac{e^{(3X+2)|t|}}{1-2|\varepsilon||t|\,e^{3X|t|}},$$

showing the theorem.

□

Theorem 8.3 implies that (8.1) is an asymptotic series in the powers of ε for the solution of (7.1), (7.2) on bounded subsets of $\mathbb{R}_t \times \mathbb{R}_x$, provided that ε is small enough. Let us express T as function of ε and x from the inequality $\varepsilon \leq (2Te^{3XT})^{-1}$ entering into the theorem. This will enable us to evaluate the characteristic times of applicability of the classical approximation corresponding to $\varepsilon = 0$.

Corollary 8.4. *Let $X/\varepsilon \gg 1$. Then $F_{\rm cl}$ approximates $F(t,x,\varepsilon)$ for small ε if $t \leq T$ with*

$$T \sim \frac{1}{6X}\,\log\frac{X}{\varepsilon}.$$

Proof. Rewrite the inequality $\varepsilon \leq (2Te^{3XT})^{-1}$ in the form

$$2\varepsilon T\,e^{3XT} \leq 1. \tag{8.8}$$

Since the left-hand side is an increasing function of $T \geq 0$, the set of all T satisfying (8.8) is an interval $[0, T_0]$, where $T_0 = T_0(X, \varepsilon)$ is the root of the equation $2\varepsilon T\, e^{3XT} = 1$.

Let us evaluate T_0. From $e^{3XT} > 1 + 3XT$ it follows that $T < (e^{3XT} - 1)/3X$ for all $T \geq 0$. Hence $T_1 < T_0 < T_2$ where T_1 and T_2 are the unique positive solutions of the equations

$$2\varepsilon\, \frac{e^{3XT_1} - 1}{3X}\, e^{3XT_1} = 1,$$
$$2\varepsilon\, T_2\, (1 + 3XT_2) = 1,$$

respectively. The solutions of these equations can be explicitly found, more precisely,

$$T_1 = \frac{1}{3X}\, \log \frac{1}{2}\left(1 + \sqrt{1 + 6\frac{X}{\varepsilon}}\right),$$
$$T_2 = \frac{1}{6X}\left(-1 + \sqrt{1 + 6\frac{X}{\varepsilon}}\right).$$

The asymptotic of T_1 in the domain of quasiclassical approach $x/\varepsilon \gg 1$ is actually

$$T_1 \sim \frac{1}{6X}\, \log \frac{X}{\varepsilon},$$

as is easy to check.

□

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